

TRANSLATING SOLITONS FOR LAGRANGIAN MEAN CURVATURE FLOW IN COMPLEX EUCLIDEAN PLANE

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ABSTRACT. Using certain solutions of the curve shortening flow, including self-shrinking and self-expanding curves or spirals, we construct and characterize many new examples of translating solitons for mean curvature flow in complex Euclidean plane. They generalize the Joyce, Lee and Tsui ones [15] in dimension two. The simplest (non trivial) example in our family is characterized as the only (non totally geodesic) Hamiltonian stationary Lagrangian translating soliton for mean curvature flow in complex Euclidean plane.

1. INTRODUCTION

The mean curvature flow (in short MCF) of an immersion $\phi : M \rightarrow \mathbb{R}^4$ of a smooth surface M is a family of immersions $F : M \times [0, \epsilon) \rightarrow \mathbb{R}^4$ parametrized by t that satisfies

$$(1) \quad \frac{d}{dt}F_t(p) = H(p, t), \quad F_0 = \phi,$$

where $H(p, t)$ is the mean curvature vector of $F_t(M)$ at $F_t(p) = F(p, t)$. The evolution of a Lagrangian surface in complex Euclidean plane \mathbb{C}^2 by its mean curvature preserves its Lagrangian character and it is called the *Lagrangian mean curvature flow*.

Some interesting problems rather far from trivial in this setting are, on the one hand, to understand the possible singularities that can occur during the flow in finite time and, on the other hand, if it is possible to show that the singularities for Lagrangian MCF are isolated. A. Neves constructed in [18] examples of Lagrangians in \mathbb{C}^2 having the Lagrangian angle as small as desired and for which the Lagrangian MCF develops a finite-time singularity. But he also proved in [18] that assuming certain properties on the initial Lagrangian surface, like almost calibrated, i.e. the oscillation of the Lagrangian angle to be strictly smaller than π , if one rescales the flow around a fixed point in space-time, connected components of this rescaled flow converge to an area-minimizing union of planes.

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In geometric flows such as the Ricci flow or the Lagrangian MCF, singularities are often locally modelled on soliton solutions, such as Lagrangians which are moved by rescaling or translation by MCF. When the evolution is a homothety we get the *self-similar solutions* for MCF. In the Lagrangian context they have been considered by several authors; see for example [4], [15], [16] and [17]. The study of this type of solutions is hoped to give a better understanding of the flow at a singularity since by Huisken's monotonicity formula [14], any central blow-up of a finite-time singularity of the mean curvature flow is a self-similar solution.

J. Chen and J. Li [7] and M.-T. Wang [22] proved independently that there is no Type I singularity along the almost calibrated Lagrangian mean curvature flow. Therefore it is of great interest to understand dilations of the flow where the point at which we center the dilation changes with the scale, called Type II dilations, which converge to an eternal solution with second fundamental form uniformly bounded. One of the most important examples of Type II singularities is a class of eternal solutions known as *translating solitons*, which are surfaces which evolve by translating in space with constant velocity.

The eternal solution F_t , $t \in \mathbb{R}$, defined by

$$(2) \quad F_t(x, y) = (-\log \cos y + t, y, x, 0), \quad -\frac{\pi}{2} < y < \frac{\pi}{2}, \quad x \in \mathbb{R}$$

is called the *grim-reaper* and it is probably the most known example of translating solution to MCF.

In [19], A. Neves and G. Tian gave conditions that exclude the existence of nontrivial translating solutions to Lagrangian MCF. More precisely, they proved that translating solutions with an L^2 bound on the mean curvature vector are planes and almost calibrated translating solutions which are static are also planes.

D. Joyce, Y.-I. Lee and M.-P. Tsui found out in [15] new surprising translating solitons for Lagrangian MCF with oscillation of the Lagrangian angle arbitrarily small. They play the same role as cigar solitons in Ricci flow and are important in studying the regularity of Lagrangian MCF. Moreover, joint to the grim-reaper (2), these examples show that the geometric conditions on the above results in [19] are optimal.

In Section 2 we describe the main geometric properties of the Lagrangian translating solitons and recall some examples. Some other interesting properties of them are studied in [11].

In Section 3 we generalize Joyce, Lee and Tsui examples to a considerable extent: It is remarked in [19] that they are associated to planar curves w in \mathbb{C} such that $w_t := \sqrt{2t}w$, for $t > 0$, is a solution to curve shortening flow in \mathbb{C} . However, our general construction is based in two families of planar curves α and ω depending on an angular parameter $\varphi \in [0, \pi)$ (see Proposition 2) that are special solutions to curve shortening flow (see Lemma 1), in the sense that their flows are a kind of composition of dilations and rotations

with suitable velocities depending on φ . For instance, in the case $\varphi = \pi/2$ we must consider α and ω spirals (i.e. travelling waves in the polar angle, see [8]) with opposite velocities; and in the case $\varphi = 0$, we require this time self-similar solutions for the curve shortening flow of opposite characters, that is, α must be a self-shrinking curve while ω must be a self-expanding one. Just when in this particular case $\varphi = 0$ we consider α as a straight line passing through the origin, we arrive at the above Joyce, Lee and Tsui examples (see Corollary 1).

In [4], the authors classified the Hamiltonian stationary Lagrangian self-similar solutions for Lagrangian mean curvature flow in complex Euclidean plane. Three one-parameter families of surfaces with different topologies (including embedded nontrivial planes) appeared. In Section 4 we characterize locally all our examples (see Theorem 1) in terms of an analytical condition on the Hermitian product of the position vector of the immersion and the translating vector that allow us *separation of variables*. As a consequence we get in Corollary 3 the classification of the Hamiltonian stationary Lagrangian translating solitons for Lagrangian mean curvature flow in complex Euclidean plane. In contrast to the self-similar case, only one example appears (see Corollary 2): a embedded complete nontrivial plane given by

$$\mathcal{M} = \{(z, w) \in \mathbb{C}^2 : w^2 = 2\operatorname{Re} z e^{-2i\operatorname{Im} z}, \operatorname{Re} z \geq 0\}.$$

It corresponds in our construction to the simplest nontrivial possible election of α (the circle $\alpha(t) = e^{it}$) and ω (the line $\omega(s) = s$) in the particular case $\varphi = 0$.

Joyce, Lee and Tsui examples are the only ones in our family with oscillation of the Lagrangian angle arbitrarily small. Therefore it would be very important to solve the open question if they can arise as blow-ups of finite time singularities for Lagrangian mean curvature flow.

2. PRELIMINARIES

2.1. Lagrangian surfaces in complex Euclidean plane. In the complex Euclidean plane \mathbb{C}^2 we consider the bilinear Hermitian product defined by

$$(z, w) = z_1 \bar{w}_1 + z_2 \bar{w}_2, \quad z, w \in \mathbb{C}^2.$$

Then $\langle \cdot, \cdot \rangle = \operatorname{Re}(\cdot, \cdot)$ is the Euclidean metric on \mathbb{C}^2 and $\omega = -\operatorname{Im}(\cdot, \cdot)$ is the Kaehler two-form given by $\omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle$, where J is the complex structure on \mathbb{C}^2 . We also consider the closed complex-valued 2-form given by $\Omega = dz_1 \wedge dz_2$ and the Liouville 1-form λ given by $d\lambda = 2\omega$.

Let $\phi : M \rightarrow \mathbb{C}^2$ be an isometric immersion of a surface M into \mathbb{C}^2 . ϕ is said to be Lagrangian if $\phi^*\omega = 0$. Then we have $\phi^*T\mathbb{C}^2 = \phi_*TM \oplus J\phi_*TM$, where TM is the tangent bundle of M . The second fundamental form σ of ϕ is given by $\sigma(v, w) = JA_{Jv}w$, where A is the shape operator, and so the trilinear form $C(\cdot, \cdot, \cdot) = \langle \sigma(\cdot, \cdot), J\cdot \rangle$ is fully symmetric.

If M is orientable and ω_M denotes the area form of M , then $\phi^*\Omega = e^{i\beta}\omega_M$, where $\beta : M \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ is called the *Lagrangian angle* map of ϕ (see [12]). In

general β is a multivalued function; nevertheless $d\beta$ is a well defined closed 1-form on M and its cohomology class is called the *Maslov class*. When β is a single valued function the Lagrangian is called *zero-Maslov class* and if $\cos \beta \geq \epsilon$ for some $\epsilon > 0$ then the Lagrangian is said to be *almost calibrated*. It is remarkable that β satisfies (see for example [21])

$$(3) \quad J\nabla\beta = H = \Delta\phi,$$

where H is the mean curvature vector of ϕ , defined by $H = \text{trace } \sigma$ and Δ is the Laplace operator of the induced metric on M .

If β is constant, say $\beta \equiv \beta_0$ or, equivalently $H = 0$, then the Lagrangian immersion ϕ is calibrated by $\text{Re}(e^{-i\beta_0}\Omega)$ and hence area-minimizing. They are referred as being *Special Lagrangian*.

A Lagrangian submanifold is called *Hamiltonian stationary* if the Lagrangian angle β is harmonic, i.e. $\Delta\beta = 0$, where Δ is the Laplace operator on M . Hamiltonian stationary Lagrangian (in short HSL) surfaces are critical points of the area functional with respect to a special class of infinitesimal variations preserving the Lagrangian constraint; namely, the class of compactly supported Hamiltonian vector fields (see [20]). Examples of HSL surfaces in \mathbb{C}^2 can be found in [2], [6] and [13].

2.2. Translating solitons for the mean curvature flow. Let $\phi : M \rightarrow \mathbb{R}^4$ be an immersion of a smooth surface M in Euclidean 4-space. In geometric flows such as the Ricci flow or the MCF, singularities are often locally modelled on soliton solutions. In the case of MCF, one type of soliton solutions of great interest are those moved by translating in the Euclidean space. We recall that they must be of the following form:

Definition 1. *An immersion $\phi : M \rightarrow \mathbb{R}^4$ is called a translating soliton for mean curvature flow if*

$$(4) \quad H = \mathbf{e}^\perp$$

for some nonzero constant vector $\mathbf{e} \in \mathbb{R}^4$, where \mathbf{e}^\perp denotes the normal projection of the vector \mathbf{e} and H is the mean curvature vector of ϕ . The 1-parameter family $F_t := \phi + t\mathbf{e}$, $t \in \mathbb{R}$, is then solution of (1) and we call \mathbf{e} a translating vector.

Any translating soliton for MCF must be a gradient soliton, that is, $\mathbf{e}^\top = \nabla f$, for some smooth function $f : M \rightarrow \mathbb{R}$, where \mathbf{e}^\top denotes the tangent projection of the vector \mathbf{e} . In fact, it is proved in [15] that $\mathbf{e}^\top = \nabla\langle\phi, \mathbf{e}\rangle$.

For Lagrangian translating solitons for MCF we point out the following properties.

Proposition 1. *Let $\phi : M \rightarrow \mathbb{C}^2$ be a Lagrangian translating soliton for mean curvature flow with translating vector \mathbf{e} and Lagrangian angle map β . Then:*

- (1) $\beta = -\langle\phi, J\mathbf{e}\rangle + \beta_0$, for some constant β_0 ;
- (2) $\Delta\beta + \langle\nabla\beta, \mathbf{e}\rangle = 0$;

$$(3) \quad \Delta \langle \phi, \mathbf{e} \rangle = |H|^2.$$

Proof. Using (3) and (4) we have that $\nabla \beta = -(J\mathbf{e})^\top$ and so $\nabla \beta + \nabla \langle \phi, J\mathbf{e} \rangle = 0$, which proves part 1. In addition, using (3) again, $\Delta \beta = -\Delta \langle \phi, J\mathbf{e} \rangle = -\langle \nabla \beta, \mathbf{e} \rangle$, which gives part 2. Finally, from (3) and (4) we deduce $\Delta \langle \phi, \mathbf{e} \rangle = \langle H, \mathbf{e}^\perp \rangle = |H|^2$, which is part 3. \square

In particular, part 1 in Proposition 1 says that a Lagrangian translating soliton for MCF is always zero-Maslov class and from part 3 we easily deduce that there are no compact Lagrangian translating solitons for MCF.

By scaling and choosing a suitable coordinate system in $\mathbb{R}^4 \equiv \mathbb{C}^2$, we can assume that $\mathbf{e} = (1, 0, 0, 0) \equiv (1, 0) \in \mathbb{C}^2$ without loss of generality.

2.3. Examples of Lagrangian translating solitons. The simplest examples of Lagrangian surfaces in \mathbb{C}^2 are usually found as product of planar curves. If we look for translating solitons for MCF in this family, we note that the *grim-reaper* F_t , $t \in \mathbb{R}$, defined in (2) can be written as

$$F_t(x, y) = (\gamma(y), x) + t(1, 0), \quad \gamma(y) = -\log \cos y + i y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}, \quad x \in \mathbb{R},$$

so γ being the graph of $-\log \cos y$ that we will call the grim-reaper curve. We can parameterize γ by arc length $s = 2 \operatorname{arctanh}(\tan y/2)$ obtaining

$$(5) \quad \gamma(s) = (\log \cosh s, 2 \operatorname{arctan}(\tanh s/2)), \quad s \in \mathbb{R}.$$

It is remarkable that the curvature κ_γ of γ verifies $\kappa_\gamma(s) = -\gamma_2'(s) = -1/\cosh s = -1/e^{\gamma_1(s)}$.

Using precisely this last property, it is an exercise to check that the product immersion

$$(6) \quad (s_1, s_2) \in \mathbb{R}^2 \longrightarrow (\gamma(s_1), \gamma(s_2)) \in \mathbb{C}^2$$

is a translating soliton for MCF with translating vector $(1, 1) \in \mathbb{C}^2$ and so

$$(7) \quad (s_1, s_2) \in \mathbb{R}^2 \longrightarrow (\gamma(s_1) + \gamma(s_2), \gamma(s_1) - \gamma(s_2)) \in \mathbb{C}^2$$

is a translating soliton for MCF with translating vector $(1, 0) \in \mathbb{C}^2$.

The translating solutions to mean curvature flow discovered by Joyce, Lee and Tsui in [15], for the case $n = 2$, are given by $\mathcal{F}_t = \mathcal{F} + t(1, 0)$ where \mathcal{F} can be described (see Section 1 in [19]) as follows: Let w be a curve in \mathbb{C} whose curvature vector \vec{k} satisfies $\vec{k} = w^\perp$. It can be chosen in such a way that the angle θ that its tangent vector makes with the x -axis has arbitrarily small oscillation. Then

$$(8) \quad \mathcal{F}(x, y) = \left(\frac{|w(y)|^2 - x^2}{2} - i\theta(y), x w(y) \right), \quad (x, y) \in \mathbb{R}^2.$$

It is still open the question posed in [15] and [19] about whether these translating solitons can arise as a blow-up of a finite time singularity for Lagrangian mean curvature flow. It would be very important to answer this question to develop a regularity theory for the flow.

3. NEW EXAMPLES OF LAGRANGIAN TRANSLATING SOLITONS FOR MCF

We start this section describing in the next Lemma a two-parameter family of curves that provides a curious solution to the curve shortening flow (CSF in short). Surprisingly some of them will be the key ingredient for our construction of new examples of Lagrangian translating solitons for MCF.

Lemma 1. *Let α be a unit speed planar curve. Assume there exist $a, b \in \mathbb{R}$, non null simultaneously, such that the curvature function κ_α of α satisfies*

$$(9) \quad \kappa_\alpha = a \langle \alpha, J\alpha' \rangle + b \langle \alpha, \alpha' \rangle$$

where $'$ denotes derivative with respect to the arc parameter of α . Then the family of curves $\alpha_t = \sqrt{2at+1} e^{i \frac{b}{2a} \log(2at+1)} \alpha$, with $2at+1 > 0$, is a solution to the curve shortening flow

$$(10) \quad \left(\frac{\partial}{\partial t} \alpha_t \right)^\perp = \vec{\kappa}_{\alpha_t}$$

such that $\alpha_0 = \alpha$. Moreover, κ_α satisfies the following o.d.e.

$$(11) \quad \kappa_\alpha \kappa_\alpha'' - \kappa_\alpha'^2 + \kappa_\alpha^2 (a + \kappa_\alpha^2) + b \kappa_\alpha' = 0.$$

Remark 1. In the limit cases $b = 0$ and $a \rightarrow 0$ we recover well known solutions to the curve shortening flow:

If $b = 0$, we have that the curvature vector of α verifies $\vec{\kappa}_\alpha = a \alpha^\perp$ and so α is a self-similar solution to CSF, contracting or expanding according to $a < 0$ or $a > 0$ respectively; the flow $\alpha_t = \sqrt{2at+1} \alpha$ is given by dilations of α in this case.

When $a \rightarrow 0$, we get now that $\vec{\kappa}_\alpha = b(J\alpha)^\perp$ and so α is a spiral (see [8]) solution to CSF with velocity $|b|$; the flow $\alpha_t = e^{ibt} \alpha$ is given by rotations of α in this other case.

Proof. Using that the normal vector to α_t is given by $J\alpha_t'/\sqrt{2at+1}$ and that $\kappa_{\alpha_t} = \kappa_\alpha/\sqrt{2at+1}$, (10) is equivalent to $\langle \frac{\partial}{\partial t} \alpha_t, J\alpha_t' \rangle = \kappa_\alpha$. It is an exercise to check that $\langle \frac{\partial}{\partial t} \alpha_t, J\alpha_t' \rangle = \text{Im}((a+ib)\overline{\alpha'}\alpha)$, which is precisely the condition satisfied by κ_α .

To prove the last part of the lemma, we define $f := \langle \alpha', \alpha \rangle$ and $g := \langle \alpha', J\alpha \rangle$ and so $\kappa_\alpha = bf - ag$. Using that $f' = 1 - \kappa_\alpha g$, $g' = \kappa_\alpha f$ and $f^2 + g^2 = |\alpha|^2$, it is only a long computation to check that κ_α satisfies (11). \square

In the next result, we make use of two families of planar curves described in Lemma 1 (taking $a = \mp \cos \varphi$ and $b = \pm \sin \varphi$ for a given $\varphi \in [0, \pi)$) in order to construct many new Lagrangian translating solitons for MCF.

Proposition 2. *Given $\varphi \in [0, \pi)$, let $\alpha = \alpha(t)$, $t \in I_1$, and $\omega = \omega(s)$, $s \in I_2$, be unit speed planar curves whose curvature vectors satisfy*

$$(12) \quad \vec{\kappa}_\alpha = -\cos \varphi \alpha^\perp + \sin \varphi (J\alpha)^\perp, \quad \vec{\kappa}_\omega = \cos \varphi \omega^\perp - \sin \varphi (J\omega)^\perp,$$

where \perp denotes normal component and I_1 and I_2 are intervals of \mathbb{R} .

Let define $\alpha * \omega : I_1 \times I_2 \subset \mathbb{R}^2 \rightarrow \mathbb{C}^2$ by

$$(13) \quad \varphi \neq \pi/2 : \quad (\alpha * \omega)(t, s) = \left(\frac{|\omega(s)|^2 - |\alpha(t)|^2}{2 \cos \varphi} \right. \\ \left. + (\tan \varphi - i)(\arg \alpha'(t) + \arg \dot{\omega}(s)), \alpha(t)\omega(s) \right)$$

and

$$(14) \quad \varphi = \pi/2 : \quad (\alpha * \omega)(t, s) = \left(\int_{t_0}^t \langle \alpha', J\alpha \rangle(x) dx \right. \\ \left. - \int_{s_0}^s \langle \dot{\omega}, J\omega \rangle(y) dy - i(\arg \alpha'(t) + \arg \dot{\omega}(s)), \alpha(t)\omega(s) \right),$$

where $'$ and $\dot{}$ denote the derivatives respect to t and s respectively, $t_0 \in I_1$ and $s_0 \in I_2$. Then $\alpha * \omega$ is a Lagrangian translating soliton for mean curvature flow with translating vector $(1, 0) \in \mathbb{C}^2$, whose induced metric is $(|\alpha|^2 + |\omega|^2)(dt^2 + ds^2)$ and its Lagrangian angle map is $\arg \alpha' + \arg \dot{\omega} + \pi + \varphi$.

Proof. The hypothesis on α and ω are clearly equivalent to

$$\kappa_\alpha = \cos \varphi \langle \alpha', J\alpha \rangle + \sin \varphi \langle \alpha', \alpha \rangle, \quad \kappa_\omega = -\cos \varphi \langle \dot{\omega}, J\omega \rangle - \sin \varphi \langle \dot{\omega}, \omega \rangle$$

respectively. Then, looking at α and ω like complex functions, (12) is equivalent to

$$(15) \quad \kappa_\alpha = \text{Im}(e^{i\varphi} \alpha' \overline{\alpha}), \quad \kappa_\omega = -\text{Im}(e^{i\varphi} \dot{\omega} \overline{\omega}).$$

For any $t_0 \in I_1$ and $s_0 \in I_2$, using (12) or (15), it is not difficult to check that the map $\alpha * \omega$ can be written, up to a translation, in the following common way for any $\varphi \in [0, \pi)$:

$$(16) \quad (\alpha * \omega)(t, s) = \left(e^{i\varphi} \left(\int_{s_0}^s \dot{\omega}(y) \overline{\omega(y)} dy - \int_{t_0}^t \alpha'(x) \overline{\alpha(x)} dx \right), \alpha(t)\omega(s) \right)$$

We denote $\Phi = \alpha * \omega$ and compute $\Phi_t = \alpha'(-e^{i\varphi} \overline{\alpha}, \omega)$ and $\Phi_s = \dot{\omega}(e^{i\varphi} \overline{\omega}, \alpha)$. Then we obtain $|\Phi_t|^2 = |\Phi_s|^2 = |\alpha|^2 + |\omega|^2$ and $\langle \Phi_t, \Phi_s \rangle = 0$. So $\alpha * \omega$ is a conformal Lagrangian immersion whose induced metric is written as $e^{2u}(dt^2 + ds^2)$, with $e^{2u} = |\alpha|^2 + |\omega|^2$. So (t^*, s^*) is a singular point of $\alpha * \omega$ if and only if $\alpha(t^*) = 0 = \omega(s^*)$.

Using that $e^{i\beta_{\alpha*\omega}} = e^{-2u} \det_{\mathbb{C}}(\Phi_t, \Phi_s)$, it is not difficult to get that the Lagrangian angle map $\beta_{\alpha*\omega}$ of $\alpha * \omega$ is given by $\beta_{\alpha*\omega} = \pi + \varphi + \arg \alpha' + \arg \dot{\omega}$. From (3) we conclude that the mean curvature vector $H_{\alpha*\omega}$ of $\alpha * \omega$ is $H_{\alpha*\omega} = e^{-2u}(\kappa_\alpha J\Phi_t + \kappa_\omega J\Phi_s)$.

On the other hand, $(1, 0)^\perp = -e^{-2u}(\text{Im}(\Phi_t, (1, 0))J\Phi_t + \text{Im}(\Phi_s, (1, 0))J\Phi_s)$ and hence (15) imply that $H_{\alpha*\omega} = (1, 0)^\perp$. \square

The conditions (12) or (15) are invariant by rotations of the curves α and ω . In the case $\varphi = \pi/2$, α and ω must satisfy $\vec{\kappa}_\alpha = (J\alpha)^\perp$ and $\vec{\kappa}_\omega = -(J\omega)^\perp$. Thus, two spirals α and ω with opposite velocities ± 1 (see Remark 1) provide, under the construction $\alpha * \omega$, a Lagrangian translating soliton for

MCF. Since $\kappa_\alpha = \langle \alpha, \alpha' \rangle$ and $\kappa_\omega = -\langle \dot{\omega}, \omega \rangle$, we get that the Lagrangian angle map in this case $\varphi = \pi/2$ is given, up to a constant, by $(|\alpha(t)|^2 - |\omega(s)|^2)/2$.

In the same direction we now emphasize the case $\varphi = 0$.

Corollary 1. *Let α and ω self-similar solutions for the curve shortening flow satisfying $\vec{\kappa}_\alpha = -\alpha^\perp$ and $\vec{\kappa}_\omega = \omega^\perp$. Then $\alpha * \omega : I_1 \times I_2 \subset \mathbb{R}^2 \rightarrow \mathbb{C}^2$ given by*

$$(17) \quad (\alpha * \omega)(t, s) = \left(\frac{|\omega(s)|^2 - |\alpha(t)|^2}{2} - i(\arg \alpha'(t) + \arg \dot{\omega}(s)), \alpha(t)\omega(s) \right)$$

is a Lagrangian translating soliton for mean curvature flow with translating vector $(1, 0) \in \mathbb{C}^2$.

By considering the straight lines $\alpha_0(t) = t$ and $\omega_0(s) = s$, the circle $\alpha_1(t) = e^{it}$, joint to self-shrinking curves α_S and self-expanding curves ω_E , we show up the following particular examples:

(i)

$$(\alpha_0 * \omega_E)(t, s) = \left(\frac{|\omega_E(s)|^2}{2} - i \arg \dot{\omega}_E(s) - \frac{t^2}{2}, t \omega_E(s) \right),$$

which correspond to the Joyce, Lee and Tsui examples such as described in (8);

(ii)

$$(\alpha_1 * \omega_E)(t, s) = \left(\frac{|\omega_E(s)|^2}{2} - i \arg \dot{\omega}_E(s) - it, e^{it} \omega_E(s) \right),$$

for which ∂_t is a Killing vector field;

(iii)

$$(\alpha_S * \omega_0)(t, s) = \left(\frac{s^2}{2} - \frac{|\alpha_S(t)|^2}{2} - i \arg \alpha'_S(t), \alpha_S(t)s \right),$$

which satisfies that its Lagrangian angle map is the angle that the tangent vector $\alpha'_S(t)$ makes with a fixed direction.

Proof. The result follows applying Proposition 2 with $\varphi = 0$ and taking into account that in the particular case (ii) the complete induced metric is given by $(1 + |\omega_E(s)|^2)(dt^2 + ds^2)$ and in the particular case (iii) the Lagrangian angle map is, up to a constant, the argument of $\alpha'_S(t)$. \square

In Lemma 10.39 of [9] it is proved that any complete self-shrinking planar curve is either a straight line passing through the origin or it lies in a bounded set. The self-shrinking curves found out by Abresh and Langer in [1] include a countable family of non embedded closed curves. However, the self-expanding planar curves ω_E are embedded and have two ends asymptotic to two straight lines (see for example [3] or [10]).

The totally geodesic Lagrangian plane is easily recovered in the above construction by $(\alpha_0 * \omega_0)(t, s) = \left(\frac{s^2 - t^2}{2}, ts \right)$. If we finally consider the example $\alpha_1 * \omega_0$, we get the following result.

Corollary 2. *Let define $\Phi : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ by*

$$\Phi(t, s) = \left(\frac{s^2}{2} - it, e^{it}s \right).$$

Then Φ is a Hamiltonian stationary complete embedded Lagrangian translating soliton for mean curvature flow with translating vector $(1, 0) \in \mathbb{C}^2$. In addition, $\Phi(\mathbb{R}^2) = \mathcal{M} := \{(z, w) \in \mathbb{C}^2 : w^2 = 2\operatorname{Re} z e^{-2i\operatorname{Im} z}, \operatorname{Re} z \geq 0\}$.

Proof. We observe that $\Phi = \alpha_1 * \omega_0$. So, it is clear that its induced metric is $(1 + s^2)(dt^2 + ds^2)$ and its Lagrangian angle is $\beta(t) = 3\pi/2 + t$. Then $\Delta\beta = 0$ and so Φ is Hamiltonian stationary.

Finally, it is clear that $\Phi(\mathbb{R}^2) \subset \mathcal{M}$. Given $(z, w) \in \mathcal{M}$, we take $t = -\operatorname{Im} z$ and $s = w e^{i\operatorname{Im} z}$. Since $s^2 = 2\operatorname{Re} z \geq 0$, s is well defined and it is easy to check that $\Phi(t, s) = (z, w)$. \square

4. CLASSIFICATION OF SEPARABLE LAGRANGIAN TRANSLATING SOLITONS

In this section we characterize locally the examples of Lagrangian translating solitons introduced in Proposition 2 under a hypothesis that will allow us to separate variables in the integration of the equations that translate (4).

Theorem 1. *Let $\phi : M^2 \rightarrow \mathbb{C}^2$ be a Lagrangian translating soliton for mean curvature flow with translating vector \mathbf{e} . Assume that there exists a local isothermal coordinate $z = x + iy$ such that the smooth complex function (ϕ, \mathbf{e}) satisfies $\frac{\partial^2}{\partial x \partial y}(\phi, \mathbf{e}) = 0$. Then ϕ is -up to dilations- locally congruent to some of the following:*

- (i) *the product of a grim-reaper curve (5) and a straight line;*
- (ii) *the product of two grim-reaper curves (see (6) or (7));*
- (iii) *the example $\alpha \star \omega$ described in Proposition 2 for some $\varphi \in [0, \pi)$.*

Proof. We start considering the translating vector $\mathbf{e} = (1, 0) \in \mathbb{C}^2$ without restriction and denoting $F = \langle \phi, \mathbf{e} \rangle$ and $G = \langle \phi, J\mathbf{e} \rangle$. Thus $\phi = (F + iG, \psi)$, where $\psi : M \rightarrow \mathbb{C}$ is the second component of ϕ . We will work in a local isothermal coordinate $z = x + iy$ on M such that the induced metric, also denoted by $\langle \cdot, \cdot \rangle$, is written as $\langle \cdot, \cdot \rangle = e^{2u}|dz|^2$ with $|dz|^2$ the Euclidean metric. So we have that

$$(18) \quad F_x^2 + G_x^2 + |\psi_x|^2 = e^{2u} = F_y^2 + G_y^2 + |\psi_y|^2, \quad F_x F_y + G_x G_y + \langle \psi_x, \psi_y \rangle = 0$$

and the Lagrangian character leads to

$$(19) \quad F_y G_x - F_x G_y + \langle \psi_x, J\psi_y \rangle = 0.$$

Using (18) and (19), taking into account that ψ_x and ψ_y are both vectors in \mathbb{C} , it is not difficult to get that

$$(20) \quad e^{2u} = F_x^2 + G_x^2 + F_y^2 + G_y^2, \quad |\psi_x|^2 = F_y^2 + G_y^2, \quad |\psi_y|^2 = F_x^2 + G_x^2.$$

From Proposition 1, (3) and (4) we also deduce that F and G must satisfy

$$(21) \quad F_{xx} + F_{yy} = G_x^2 + G_y^2, \quad G_{xx} + G_{yy} = -F_x G_x - F_y G_y$$

and ψ verifies

$$(22) \quad \psi_{xx} + \psi_{yy} = -G_x J\psi_x - G_y J\psi_y.$$

From now on, by the hypothesis on separability, we can assume that the isothermal coordinate we are working in satisfies $(\phi, (1, 0))_{xy} = 0$. This means nothing but $F_{xy} = 0 = G_{xy}$. We remark that adding a constant to F or G produces a congruent immersion. We make use of these two conditions in the following.

On the one hand, there exist smooth real functions $\xi = \xi(x)$ and $\theta = \theta(y)$ such that

$$G(x, y) = -(\xi(x) + \theta(y)).$$

Then we consider planar curves $\alpha = \alpha(x)$, $x \in I_1 \subset \mathbb{R}$, and $\omega = \omega(y)$, $y \in I_2 \subset \mathbb{R}$, arc length parameterized whose curvature functions are given by $\kappa_\alpha(x) = \xi'(x)$ and $\kappa_\omega(y) = \theta'(y)$ respectively. Up to rotations, we can write

$$(23) \quad \alpha'(x) = e^{i\xi(x)}, \quad \dot{\omega}(y) = e^{i\theta(y)}$$

and, up to a constant, we can also write

$$(24) \quad G(x, y) = -\arg \alpha'(x) - \arg \dot{\omega}(y) = -\int \kappa_\alpha(x) dx - \int \kappa_\omega(y) dy.$$

We also remark that, according to Proposition 1(1), the Lagrangian angle map β of ϕ is given by

$$\beta(x, y) = \xi(x) + \theta(y) + \beta_0, \quad \beta_0 \in \mathbb{R}.$$

On the other hand, there exist smooth real functions $A = A(x)$ and $B = B(y)$ such that

$$(25) \quad F(x, y) = A(x) + B(y).$$

Putting (25) and (24) in (21), we can find $\lambda, \mu \in \mathbb{R}$ such that A and B must satisfy the following ordinary differential equations:

$$(26) \quad \kappa_\alpha A' = \mu - \kappa'_\alpha, \quad A'' = \kappa_\alpha^2 - \lambda,$$

$$(27) \quad \kappa_\omega \dot{B} = -\mu - \dot{\kappa}_\omega, \quad \ddot{B} = \kappa_\omega^2 + \lambda.$$

We notice that the o.d.e.'s for A and B are the same interchanging the pair (λ, μ) by $(-\lambda, -\mu)$. Let us study (26) for example. If $\kappa_\alpha \equiv 0$ then $\mu = 0$ and $A(x) = -\lambda x^2/2 - b_1 x$, with $b_1 \in \mathbb{R}$, up to a translation. If κ_α is non null, outside the zeroes of κ_α , we get $A(x) = -\log |\kappa_\alpha(x)| + \mu \int dx/\kappa_\alpha(x)$, where κ_α is a solution to

$$(28) \quad \kappa_\alpha \kappa_\alpha'' - \kappa_\alpha'^2 + \mu \kappa_\alpha' = \kappa_\alpha^2 (\lambda - \kappa_\alpha^2).$$

By the above observation, analogously if $\kappa_\omega \equiv 0$ then $\mu = 0$ and $B(y) = \lambda y^2/2 + b_2 y$, with $b_2 \in \mathbb{R}$, up to a translation. If κ_ω is non null, outside

the zeroes of κ_ω , we get $B(y) = -\log |\kappa_\omega(y)| - \mu \int dy/\kappa_\omega(y)$, where κ_ω is a solution to

$$(29) \quad \kappa_\omega \dot{\kappa}_\omega - \dot{\kappa}_\omega^2 - \mu \dot{\kappa}_\omega = \kappa_\omega^2(-\lambda - \kappa_\omega^2).$$

Hence we are devoted to study the o.d.e.'s (28) and/or (29) in the following lemma, which deserves interest by itself. We recognize (28) and (29) in Lemma 1 taking $a = \mp\lambda$ and $b = \pm\mu$ respectively.

Lemma 2. *Given $\lambda, \mu \in \mathbb{R}$, consider the ordinary differential equation*

$$(30) \quad k\ddot{k} - \dot{k}^2 + k^2(\lambda + k^2) = \mu\dot{k}.$$

- If $(\lambda, \mu) = (0, 0)$, then $\dot{k}^2/k^2 + k^2 = \rho^2 \geq 0$ is a first integral of (30) and $k(y) = \rho/\cosh(\rho y)$, $y \in \mathbb{R}$, is its solution satisfying $\dot{k}(0) = 0$.
- If $(\lambda, \mu) \neq (0, 0)$, let k_w be the curvature of a unit speed planar curve w in \mathbb{C} satisfying $k_w = -\lambda\langle\dot{w}, Jw\rangle - \mu\langle\dot{w}, w\rangle$. Then k_w is the general solution of (30). Moreover, k_w verifies:

- (1) $\frac{(\dot{k}_w + \mu)^2}{k_w^2} + k_w^2 = (\lambda^2 + \mu^2)|w|^2$,
- (2) $-\log|k_w| - \int \mu/k_w - i \int k_w = (\lambda + i\mu) \int \dot{w}\bar{w}$,
outside the zeroes of k_w .

Proof of Lemma 2: The case $(\lambda, \mu) = (0, 0)$ is an exercise. When $(\lambda, \mu) \neq (0, 0)$, it was proved in Lemma 1 that k_w satisfies (30). We define again $f := \langle\dot{w}, w\rangle$ and $g := \langle\dot{w}, Jw\rangle$ and so $k_w = -\lambda g - \mu f$. Using that $\dot{f} = 1 - k_w g$, $\dot{g} = k_w f$ and $f^2 + g^2 = |w|^2$, it is straightforward to check that k_w satisfies part (1) in the Lemma. To prove part (2), we observe that $(-\log|k_w| - \int \mu/k_w - i \int k_w)' = (\lambda + i\mu)(f + ig) = (\lambda + i\mu)\dot{w}\bar{w}$. Finally, given arbitrary initial conditions $k_0 = k(0)$ and $k_1 = \dot{k}(0)$ for (30), the system of equations $-\lambda g(0) - \mu f(0) = k_0$, $\mu g(0) - \lambda f(0) = \mu + k_1$ has an unique solution since $(\lambda, \mu) \neq (0, 0)$. This shows that k_w is the general solution of (30) and concludes the proof of Lemma 2.

We now proceed to integrate $\phi = (F + iG, \psi)$ collecting first the information from (25), (24) and (20). According to the above discussion, we must distinguish the following cases:

Case (i): $\kappa_\alpha \equiv 0 \equiv \kappa_\omega$. In particular $\mu = 0$ and G is constant. Hence β is constant too and so ϕ is minimal. Moreover, we have that

$$F(x, y) = -\lambda x^2/2 - b_1 x + \lambda y^2/2 + b_2 y$$

and

$$e^{2u(x, y)} = (\lambda x + b_1)^2 + (\lambda y + b_2)^2$$

Case (ii): $\kappa_\alpha \equiv 0$ and κ_ω non null. In particular $\mu = 0$. We now get that

$$F(x, y) = -\lambda x^2/2 - b_1 x - \log|\kappa_\omega(y)|, \quad G(y) = -\int \kappa_\omega(y) dy$$

and

$$e^{2u(x,y)} = (\lambda x + b_1)^2 + \dot{\kappa}_\omega(y)^2 / \kappa_\omega(y)^2 + \kappa_\omega(y)^2,$$

where κ_ω is a solution of (29) with $\mu = 0$.

Case (iii): κ_α non null and $\kappa_\omega \equiv 0$. In particular $\mu = 0$. Analogously we get that

$$F(x, y) = -\log |\kappa_\alpha(x)| + \lambda y^2 / 2 + b_2 y, \quad G(x) = -\int \kappa_\alpha(x) dx$$

and

$$e^{2u(x,y)} = \kappa'_\alpha(x)^2 / \kappa_\alpha(x)^2 + \kappa_\alpha(x)^2 + (\lambda y + b_2)^2,$$

where κ_α is a solution of (28) with $\mu = 0$.

Case (iv): κ_α and κ_ω both non null. We arrive at

$$F(x, y) = -\log |\kappa_\alpha(x)| + \mu \int dx / \kappa_\alpha(x) - \log |\kappa_\omega(y)| - \mu \int dy / \kappa_\omega(y),$$

$$G(x, y) = -\int \kappa_\alpha(x) dx - \int \kappa_\omega(y) dy$$

and

$$e^{2u(x,y)} = (\kappa'_\alpha(x) - \mu)^2 / \kappa_\alpha(x)^2 + \kappa_\alpha(x)^2 + (\dot{\kappa}_\omega(y) + \mu)^2 / \kappa_\omega(y)^2 + \kappa_\omega(y)^2,$$

where κ_α is a solution of (28) and κ_ω is a solution of (29).

In order to use Lemma 2 we analyze the two given possibilities. First we fix $(\lambda, \mu) \neq (0, 0)$. Using Lemma 2, we know that ω and α must satisfy $\kappa_\omega = -\lambda \langle \dot{\omega}, J\omega \rangle - \mu \langle \dot{\omega}, \omega \rangle$ and $\kappa_\alpha = \lambda \langle \alpha', J\alpha \rangle + \mu \langle \alpha', \alpha \rangle$ and, in addition, up to a constant we have that

$$(31) \quad (F + iG)(x, y) = (\lambda + i\mu) \left(\int \dot{\omega}(y) \bar{\omega}(y) dy - \int \alpha'(x) \bar{\alpha}(x) dx \right)$$

and

$$(32) \quad e^{2u(x,y)} = (\lambda^2 + \mu^2)(|\alpha(x)|^2 + |\omega(y)|^2).$$

In the cases (i), (ii) and (iii), necessarily $\lambda \neq 0$ since $\mu = 0$. If we make changes of parameters $(x \rightarrow x + b_1/\lambda, y \rightarrow y + b_2/\lambda)$ then (31) and (32) also hold (up to a translation) considering $\alpha(x) = x$ and $\omega(y) = y$ when $\kappa_\alpha \equiv 0$ and $\kappa_\omega \equiv 0$ respectively.

Moreover, it is not difficult to get, taking into account (20), (19) and (32), that

$$(33) \quad |\psi_x|^2 = (\lambda^2 + \mu^2)|\omega|^2, \quad |\psi_y|^2 = (\lambda^2 + \mu^2)|\alpha|^2, \quad (\psi_x, \psi_y) = (\lambda^2 + \mu^2)\alpha' \bar{\alpha} \bar{\omega} \omega.$$

Analyzing (22) after considering (24), using (23) and (33), we conclude that there exist two complex functions $c_i = c_i(x, y)$, $i = 1, 2$, such that

$$(34) \quad \psi_x = c_1 \alpha', \quad \psi_y = c_2 \dot{\omega}, \quad (c_1)_x \alpha' + (c_2)_y \dot{\omega} = 0, \quad \alpha c_1 = \omega c_2.$$

Since $|c_1|^2 = |\psi_x|^2$ and $|c_2|^2 = |\psi_y|^2$, from (34) we can find two real functions $\nu_i = \nu_i(x, y)$, $i = 1, 2$, in order to write $c_1 = \sqrt{\lambda^2 + \mu^2} |\omega| e^{i\nu_1}$ and $c_2 = \sqrt{\lambda^2 + \mu^2} |\alpha| e^{i\nu_2}$. The last two equations of (34) translate into

$$(35) \quad |\omega|(\nu_1)_x \alpha' e^{i\nu_1} + |\alpha|(\nu_2)_y \dot{\omega} e^{i\nu_2} = 0, \quad |\omega| \alpha e^{i\nu_1} = |\alpha| \omega e^{i\nu_2},$$

which lead to $(\nu_1)_x \omega \alpha' + (\nu_2)_y \dot{\omega} \alpha = 0$. As α and α' (resp. ω and $\dot{\omega}$) are necessarily linearly independent in this case, we deduce that $(\nu_1)_x = 0 = (\nu_2)_y$ and hence there is a constant ν_0 such that $|\omega| e^{i\nu_1} / \omega = |\alpha| e^{i\nu_2} / \alpha = e^{i\nu_0}$ thanks to the last equation in (35). Using the first two equations of (34), we arrive at $\psi_x = \sqrt{\lambda^2 + \mu^2} e^{i\nu_0} \alpha' \omega$ and $\psi_y = \sqrt{\lambda^2 + \mu^2} e^{i\nu_0} \alpha \dot{\omega}$. Thus, up to a rotation and a translation, we finally get that

$$(36) \quad \psi(x, y) = \sqrt{\lambda^2 + \mu^2} \alpha(x) \omega(y).$$

Therefore we conclude from (31) and (36) that

$$\phi(x, y) = \left((\lambda + i\mu) \left(\int \dot{\omega}(y) \bar{\omega}(y) dy - \int \alpha'(x) \bar{\alpha}(x) dx \right), \sqrt{\lambda^2 + \mu^2} \alpha(x) \omega(y) \right)$$

where α and ω satisfy $\kappa_\omega = -\lambda \langle \dot{\omega}, J\omega \rangle - \mu \langle \dot{\omega}, \omega \rangle$ and $\kappa_\alpha = \lambda \langle \alpha', J\alpha \rangle + \mu \langle \alpha', \alpha \rangle$. Up to dilations, there is no restriction taking $\lambda + i\mu = e^{i\varphi}$, with $\varphi \in [0, 2\pi)$. So this is exactly the common expression (16) for the examples $\alpha \star \omega$ introduced in Proposition 2. Interchanging the roles of α and ω , it is enough to consider $\varphi \in [0, \pi)$. The conclusion is that ϕ is one of the examples mentioned in part (iii) of the statement of Theorem 1.

We finally study the remaining case $(\lambda, \mu) = (0, 0)$. We remark that in cases (i), (ii) and (iii) we only have to consider $\lambda = 0$ since μ was necessary zero there.

In case (i), $\lambda = 0$ implies that u is constant and so the immersion is flat besides minimal. Thus it is totally geodesic. Recall that $\alpha_0 \star \omega_0$ recovers a totally geodesic Lagrangian plane.

In case (ii), following Lemma 2, up to a constant we get that

$$(F + iG)(x, y) = -b_1 x + \log \cosh(\rho y) - i\rho \int \frac{dy}{\cosh \rho y}.$$

In the coordinates $(t, s) = (-\rho(x, y))$ and putting $-b_1/\rho = \sinh \delta$, $\delta \in \mathbb{R}$, we rewrite

$$(F + iG)(t, s) = -\sinh \delta t + \log \cosh s + i \int \frac{ds}{\cosh s} = -\sinh \delta t + \gamma(s),$$

where $\gamma(s)$ (see (5)) is just the graph $(-\log v, v)$, $v \in (-\pi/2, \pi/2)$, parameterized by arc length. A similar study of (22) like in the previous case using now in (18), (19) and (20) the above expressions of F and G leads to

$$\psi(t, s) = t + \sinh \delta \gamma(s).$$

Then we get that $\phi(t, s) = A(\gamma(s), t)$, where A is the matrix $\begin{pmatrix} 1 & -\sinh \delta \\ \sinh \delta & 1 \end{pmatrix}$.

Thus we arrive at (i) in the statement of Theorem 1.

Case (iii) is completely analogous to case (ii) changing ω by α and b_1 by $-b_2$ so that we get the same conclusion.

In case (iv), applying twice Lemma 2 and the same argument that in case (ii), we deduce that

$$(F + iG)(s_1, s_2) = \gamma(s_1) + \gamma(s_2), \quad \psi(s_1, s_2) = \gamma(s_1) - \gamma(s_2).$$

Hence we arrive at (ii) in the statement of Theorem 1. \square

Corollary 3. *Let $\phi : M \rightarrow \mathbb{C}^2$ be a Hamiltonian stationary (non totally geodesic) Lagrangian translating soliton for mean curvature flow. Then $\phi(M)$ is -up to dilations- an open subset of the Lagrangian \mathcal{M} given in Corollary 2.*

Proof. We follow the same election for the translating vector and use the same notation that at the beginning of the proof of Theorem 1. We can associate to any Lagrangian immersion $\phi : M \rightarrow \mathbb{C}^2$ a differential form Υ on M (see [5]) defined by

$$\Upsilon(z) = \bar{h}(z)dz, \text{ with } h(z) = \omega(\partial_{\bar{z}}, H),$$

where $z = x + iy$ is a local isothermal coordinate on M and ω is extended \mathbb{C} -linearly to the complexified tangent bundles. Then (3) translates into $h = \beta_{\bar{z}}$, with β the Lagrangian angle map of ϕ , and the Coddazi equation of ϕ gives (see [5]) $\text{Im}(h_z) = 0$.

Thus $\bar{h}_{\bar{z}} = h_z = \beta_{z\bar{z}} = 0$ since β is harmonic because ϕ is Hamiltonian stationary. Hence Υ is a holomorphic differential and we can normalize $h \equiv -1/2$.

Using Proposition 1.(1), we have that also $h = -G_{\bar{z}}$ and so $G_x \equiv -1$ and $G_y \equiv 0$ after the above normalization. In particular, $G_{xy} = 0$. Looking at the second equation of (21) we easily deduce that $F_x = 0$ and then $F_{xy} = 0$. We have proved that ϕ verifies the hypothesis of Theorem 1 and necessarily must be one the examples $\alpha \star \omega$ associated to certain $\varphi \in [0, \pi)$. We know from Proposition 2 that its induced metric is conformal and, using the expression of its Lagrangian angle map, we get that $\alpha \star \omega$ is Hamiltonian stationary if and only if $\kappa'_\alpha + \kappa_\omega = 0$. Using (11) we obtain that $\kappa_\alpha \equiv c_1 \in \mathbb{R}$ and $\kappa_\omega \equiv c_2 \in \mathbb{R}$ such that

$$(37) \quad c_1^2(c_1^2 - \cos \varphi) = 0 = c_2^2(c_2^2 + \cos \varphi).$$

If $c_1 = 0$, α must be a line and this implies that $\varphi = 0$ and, following the notation of Corollary 1, $\alpha = \alpha_0$. Using now (37) we have that $c_2 = 0$ and a similar reasoning gives that $\omega = \omega_0$. In this case, ϕ corresponds to a totally geodesic Lagrangian plane.

And if $c_1 \neq 0$, from (37) it follows that $c_1^2 = \cos \varphi$, $0 \leq \varphi < \pi/2$, and $c_2 = 0$. This last implies that ω must be a line, $\varphi = 0$ and $\omega = \omega_0$. Thus $c_1 = 1$ and we finally deduce that $\alpha = \alpha_1$. Therefore we arrive at the example $\Phi = \alpha_1 \star \omega_0$ and Corollary 2 finishes the proof. \square

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